NORTH SYDNEY BOYS HIGH SCHOOL

2024 YEAR 12 HSC ASSESSMENT TASK 3

Mathematics Advanced

General Instructions

- Reading time – 10 minutes
- Working time – **3 hours**
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Attempt all questions

Class Teacher:
(Please tick or highlight)

- ○ Ms Sarofim
- ○ Ms Cai
- ○ Ms Moss
- ○ Mr Ireland

Student Number:

(To be used by the exam markers only.)

Question No	1-10	11- 30	Total
Mark	$\frac{\square}{10}$	$\frac{\square}{90}$	$\frac{\square}{100}$

Section I

10 marks

Allow about 15 minutes for this section. Use the multiple-choice answer sheet

1 A cosine function f has a maximum value of 2, a minimum value of -1, and a period of $\frac{2}{5}$. The function could be:

A $f(x) = 3\cos(5\pi x)$

B $f(x) = 1.5\cos(\frac{4\pi x}{5})$

C $f(x) = 3\cos\left(\frac{2\pi x}{5}\right) - 0.5$

D $f(x) = 1.5\cos(5\pi x) + 0.5$

2 A particle, initially at the origin, moves in a straight line with velocity $v = 8 - 4t$ metres per second. What is the total distance the particle travels in the first 4 seconds?

A $0\ m$

B $16\ m$

C $8\ m$

D $12\ m$

3 In an arithmetic series, the sum of the first three terms is 21 and the sum of the first four terms is 36. What is the sum of the first five terms?

A 15

B 51

C 55

D 57

4 Which of the following is the range of the function $y = \frac{1}{2} - \frac{1}{2}\cos 2x$?

A $-\frac{1}{2} \leq y \leq 0$

B $-\frac{1}{2} \leq y \leq \frac{1}{2}$

C $0 \leq y \leq \frac{1}{2}$

D $0 \leq y \leq 1$

5 The number N of flies at the Longueville rubbish dump after t weeks is given by the formula $N = N_0 e^{0.04\,t}$, where N_0 is the initial number of flies.

Which expression represents the number of weeks it takes until the rubbish dump has doubled its population of flies?

A $\frac{2\,ln\,100}{5}$

B $\frac{25}{ln\,2}$

C $2\,ln\,25$

D $25\,ln\,2$

6 Which expression gives $\int tan^2x\;\;dx$?

A $sec^2x + c$

B $2\,tan\,x + c$

C $tan\,x - x + c$

D $2\,tan\,x - x + c$

7 Jordie is installing a security keypad on the door to the server room. It has the digits $0 - 9$ and codes can be created which have either 4, 5 or 6 digits.

How many more codes are available if he uses a 6 digit code rather than a 4 digit code?

A 9900

B 521 441

C 524 880

D 990 000

8 Let X and Y be independent events where $P(X) = 0 \cdot 2$ and $P(Y) = 0 \cdot 52$

What is the value of $P(\bar{X} \cap \bar{Y})$?

A $0 \cdot 616$

B $0 \cdot 72$

C $0 \cdot 384$

D $0 \cdot 28$

9 Let $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x-1}$. Which is the range of $g\big(f(x)\big)$?

A $[1, \infty]$

B $[0, 1) \cup (1, \infty)$

C $(-\infty, -1] \cup (0, \infty)$

D $(0, \infty)$

10 For what values of *x* is the curve $y = x^4 - 12x^2$ concave up?

A $x < -\sqrt{2}\ or\ x > \sqrt{2}$

B $x < -\sqrt{6}\ or\ x > \sqrt{6}$

C $-\sqrt{6} < x < \sqrt{6}$

D $-\sqrt{2} < x < \sqrt{2}$

Section II

90 marks

Question 11 (2 marks)

Solve the equation $|2x - 3| = 18$ **2**

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Question 12 (2 marks)

Solve the equation $\tan(2x) = \sqrt{3}$ in the domain $0 \leq x \leq \pi$. **2**

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Question 13 (2 marks)

A standard pack of cards consists of four suits (Diamonds, Hearts, Clubs and Spades) with 13 cards in each suit. If four cards are chosen at random from the pack and placed side by side on a table, what is the probability that they are all from different suits? **2**

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Question 14 (2 marks)

Volodymyr likes Tic Tacs(R). On Monday he eats 2, on Tuesday he eats 4, on Wednesday he eats 6, and so on in this pattern. After how many days will he have eaten a total of 132 Tic Tacs ? **2**

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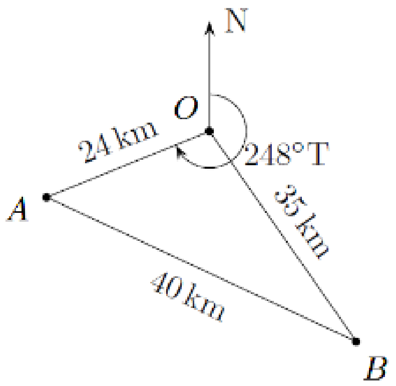
Question 15 (5 marks)

A section of land has been earmarked for a new national park. The shape of the land is shown below.



The bearing of landmark A from landmark O is 248T and the distance between them is 24km. The distance between landmark A and point B is 40km, and the distance from B to O is 35km.

(i) Show that $\angle AOB = 83°$ to the nearest degree. **2**

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(ii) Calculate the area of the planned national park, correct to the nearest square km. **2**

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(iii) What is the bearing of O from B ? **1**

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Question 16 (5 marks)

(i) Find the integral $\int \frac{10x}{x^2-8}\, dx$ **2**

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(ii) Find the exact value of $\int_2^6 2^{x-1}\, dx$ **3**

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Question 17 (2 marks)

Find $\frac{d}{dx}\ \ln\frac{x+1}{x-2}$ **2**

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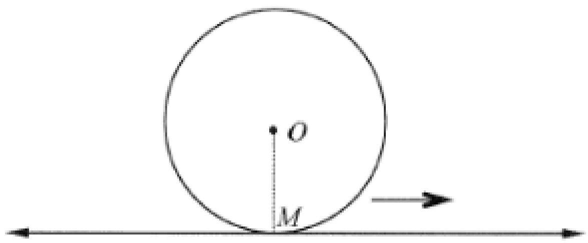
Question 18 (4 marks)

A train wheel of radius 40 cm and centre *O* rolls along a horizontal track (as shown below).

M is a point on the wheel where the wheel touches the track before it starts to roll.



(i) Through what angle does *M* rotate about *O* (in radians) after the wheel rolls 1 metre ? **1**

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(ii) After the wheel rolls 1 metre, what is the vertical height of *M* above the track? (answer to the nearest cm) **3**

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Question 19 (5 marks)

In Glenrowan, the probability that it rains on any Monday is $0 \cdot 21$. If it rains on Monday, then the probability that it rains on Tuesday is $0 \cdot 83$. If it does not rain on Monday, then the probability of rain on Tuesday is $0 \cdot 3$.

(i) Draw a probability tree diagram to show this scenario. **2**

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(ii) Calculate the probability that it rains on a Tuesday in any given week. **1**

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(iii) Given that it rains on a Tuesday, find the chance it did not rain on the Monday. **2**

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Question 20 (6 marks)

(a) Sketch the curve $y = 4e^{-2x}$ **2**

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(b) Consider the series $2e^{x} + 8e^{-x} + 32e^{-3x} + \cdots$

(i) Show that this series is geometric **1**

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(ii) Find the values of *x* for which this series has a limiting sum. **2**

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(ii) Find the limiting sum of this series in terms of *x*. **1**

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Question 21 (6 marks)

(i) Show that the curves $y = \sin 2x$ and $y = \cos x$ meet at the point where $x = \frac{\pi}{6}$. **1**

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(ii) Sketch, on the same set of axes, the curves $y = \sin 2x$ and $y = \cos x$ in the domain $0 \leq x \leq \frac{\pi}{2}$ and shade the region bounded only by those curves. **2**

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(iii) Find the area of this shaded region. **3**

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Question 22 (2 marks)

Evaluate $\sum_{r=1}^{63} \frac{1}{\sqrt{r+1}+\sqrt{r}}$ **2**

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Question 23 (5 marks)

A tank contains 200 litres of petrol. More petrol is pumped into the tank for 20 minutes, until it is full.

The volume flow rate *R* of petrol, in litres per minute, is given by

$$R \quad = 4\,(20 - t\ )$$

(i) Find a formula for the volume *V* of petrol in the tank after *t* minutes, where $t \leq 20$. **2**

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(ii) How many litres of petrol were in the tank when it was full? **1**

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(iii) How long did it take to half fill the tank? (answer in minutes) **2**

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Question 24 (2 marks)

If $log_a(xy^3) = 1$ and $log_a(x^2y) = 1$ what is the value of $log_a(xy)$? **2**

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Question 25 (3 marks)

Isis and Layla catch a train to school each day but they get on at different stations.

The probability that Isis gets a seat each morning is $\frac{1}{8}$.

The probability that Layla gets a seat each morning is $\frac{1}{21}$.

What is the probability that, on the next three mornings, at least one girl gets a seat? **3**

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Question 26 (8 marks)

Consider the curve $y = x^3 + 3x^2 - 9x - 5$.

(i) Find any stationary points and determine their nature. **4**

(ii) Sketch the curve in the domain $-5 \leq x \leq 3$ **2**
(inflection and x-intercepts are not required)

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(iii) By drawing an appropriate line on your graph, or otherwise, solve

$$x^3 + 3x^2 - 9x + 5 = 0$$ **2**

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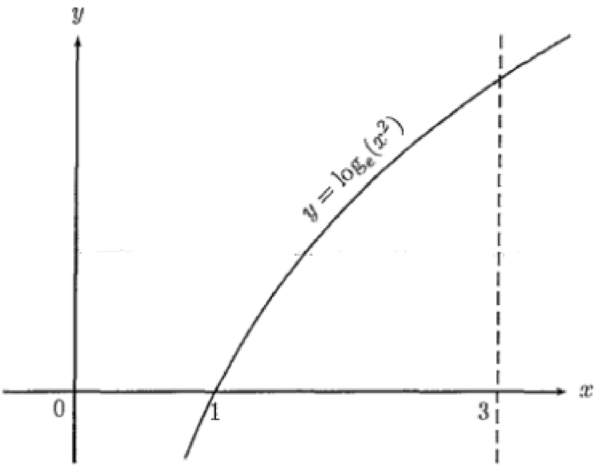
Question 27 (6 marks)

The graph of the function $y = log_e(x^2)$ is shown below:



(i) Use the Trapezoidal Rule with 5 function values to approximate $\int_1^3 log_e(x^2)\,dx$ to 3 significant figures, and explain why this approximation underestimates the value of the integral. **3**

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(ii) Find $\int_0^{\ln 9} e^{\frac{y}{2}}\, dy$ and hence find the exact value of $\int_1^3 log_e(x^2)\, dx$ **3**

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Question 28 (2 marks)

Find $f'(x)$ if it is known that $\frac{d}{dx}\,[f(2x)] = x^2$ **2**

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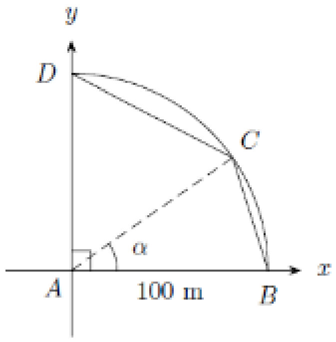
Question 29 (8 marks)

ABCD is a quadrilateral inscribed in a quarter of a circle centred at *A*, radius 100 m.

The points *B* and *D* lie on the *x* and *y* axes and the point *C* moves on the circle such that

$\angle CAB = \alpha$ as shown in the diagram below.



(i) Solve the equation $\sin(x + 15°) = \cos 24°$ **1**

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(ii) Show that the area of the quadrilateral ABCD can be expressed as **3**

$$A = 5000\,(\sin\alpha + \cos\alpha)$$

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(iii) Using calculus, show that the maximum area of this quadrilateral is $5000\sqrt{2}\ m^2$ **4**

Question 30 (3 marks)

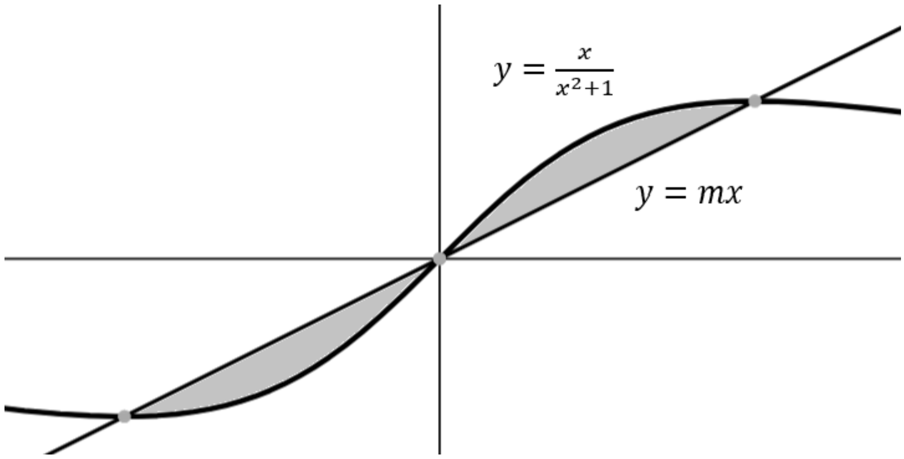
Let $f(x) = x^2 - 4x$.

Sketch the graph of $y = 2f(1 - x) + 6$, showing the location of the vertex and the intercepts. **3**

Question 31 (5 marks)

The function $y = \frac{x}{x^2+1}$ is shown below. It is an odd function.

(i) For what values of m do the line $y = mx$ and the curve $y = \frac{x}{x^2+1}$ enclose a region? **2**



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(ii) Find the area of the region (in terms of m). **3**

Question 32 (5 marks)

(i) Show that $e^x \geq 1 + x$ if $x \geq 0$. **2**

(Hint: show that the function $f(x) = e^x - (1 + x)$ is increasing for $x \geq 0.$)

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(ii) Deduce that $\frac{4}{3} \leq \int_0^1 e^{x^2}\, dx \leq e$ **3**

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ENDS

Section I

10 marks

Allow about 15 minutes for this section. Use the multiple-choice answer sheet

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A $f(x) = 3\cos(5\pi x)$

B $f(x) = 1.5\cos\left(\frac{4\pi x}{5}\right)$

C $f(x) = 3\cos\left(\frac{2\pi x}{5}\right) - 0.5$

D $f(x) = 1.5\cos(5\pi x) + 0.5$

2 A particle, initially at the origin, moves in a straight line with velocity $v = 8 - 4t$ metres per second. What is the total distance the particle travels in the first 4 seconds?

A $0\ m$

B $16\ m$

C $8\ m$

D $12\ m$

3 In an arithmetic series, the sum of the first three terms is 21 and the sum of the first four terms is 36. What is the sum of the first five terms?

A 15

B 51

C 55

D 57

4 Which of the following is the range of the function $y = \frac{1}{2} - \frac{1}{2}\cos 2x$?

A $-\frac{1}{2} \leq y \leq 0$

B $-\frac{1}{2} \leq y \leq \frac{1}{2}$

C $0 \leq y \leq \frac{1}{2}$

D $0 \leq y \leq 1$

5 The number N of flies at the Longueville rubbish dump after t weeks is given by the formula $N = N_0 e^{0.04\,t}$, where N_0 is the initial number of flies.

Which expression represents the number of weeks it takes until the rubbish dump has doubled its population of flies?

A $\frac{2\,ln\,100}{5}$

B $\frac{25}{ln\,2}$

C $2\,ln\,25$

(D) $25\,ln\,2$

6 Which expression gives $\int tan^2x\ dx$?

A $sec^2x + c$

B $2\,tan\,x + c$

(C) $tan\,x - x + c$

D $2\,tan\,x - x + c$

7 Jordie is installing a security keypad on the door to the server room. It has the digits $0 - 9$ and codes can be created which have either 4, 5 or 6 digits.

How many more codes are available if he uses a 6 digit code rather than a 4 digit code?

A 9900

B $521\,441$

C $524\,880$

(D) $990\,000$

8 Let X and Y be independent events where $P(X) = 0 \cdot 2$ and $P(Y) = 0 \cdot 52$

What is the value of $P(\bar{X} \cap \bar{Y})$?

A $0 \cdot 616$

B $0 \cdot 72$

(C) $0 \cdot 384$

D $0 \cdot 28$

9 Let $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x-1}$. Which is the range of $g(f(x))$?

A $[1, \infty]$

B $[0, 1) \cup (1, \infty)$

C $(-\infty, -1] \cup (0, \infty)$

D $(0, \infty)$

10 For what values of *x* is the curve $y = x^4 - 12x^2$ concave up?

A $x < -\sqrt{2}$ or $x > \sqrt{2}$

B $x < -\sqrt{6}$ or $x > \sqrt{6}$

C $-\sqrt{6} < x < \sqrt{6}$

D $-\sqrt{2} < x < \sqrt{2}$

Section II

90 marks

Question 11 (2 marks)

Solve the equation $|2x - 3| = 18$ **2**

$2x - 3 = 18$ or $2x - 3 = -18$

$\therefore$ $2x = 21$ or $2x = -15$

$\therefore$ $x = \frac{21}{2}$ or $-\frac{15}{2}$ ✓✓

Question 12 (2 marks)

Solve the equation $\tan(2x) = \sqrt{3}$ in the domain $0 \leq x \leq \pi$. **2**

We have $0 \leq 2x \leq 2\pi$

So, if $\tan 2x = \sqrt{3}$,

$\therefore$ $2x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$

$\therefore$ $x = \frac{\pi}{6}, \frac{2\pi}{3}$ ✓✓

Question 13 (2 marks)

A standard pack of cards consists of four suits (Diamonds, Hearts, Clubs and Spades) with 13 cards in each suit. If four cards are chosen at random from the pack and placed side by side on a table, what is the probability that they are all from different suits? **2**

$$P(\text{all different suits}) = \frac{52}{52} \times \frac{39}{51} \times \frac{26}{50} \times \frac{13}{49}$$ ✓

$$= \frac{685\ 464}{6\ 497\ 400}$$

$$= \frac{2197}{20825}$$

$$\doteqdot 0 \cdot 1055 \quad (4\ d.p.)$$ ✓

Question 14 (2 marks)

Volodymyr likes Tic Tacs(R). On Monday he eats 2, on Tuesday he eats 4, on Wednesday he eats 6, and so on in this pattern. After how many days will he have eaten a total of 132 Tic Tacs ? **2**

A. P. is $2, 4, 6, 8, \ldots$ where $a = 2$, $d = 2$

We have $S_n = \frac{n}{2}(2a + (n-1)d) = 132$

$\therefore \frac{n}{2}(4 + 2n - 2) = 132$

$\frac{n}{2}(2 + 2n) = 132 \quad \therefore n(1+n) = 132$ ✓

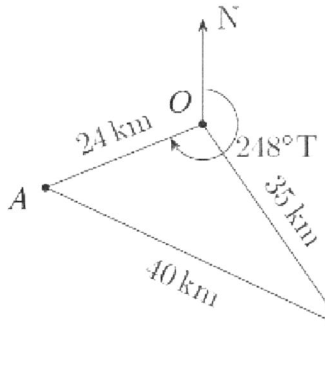
$\therefore n^2 + n - 132 = 0$

$(n+12)(n-11) = 0 \quad \therefore n = -12$ ~~or~~ $n = 11$

But $n > 0 \quad \therefore n = 11$ i.e. 11 days. ✓

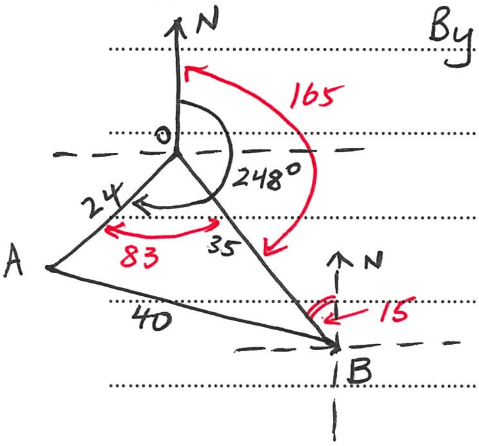
Question 15 (5 marks)

A section of land has been earmarked for a new national park. The shape of the land is shown below.



The bearing of landmark A from landmark O is 248T and the distance between them is 24km. The distance between landmark A and point B is 40km, and the distance from B to O is 35km.

(i) Show that $\angle AOB = 83°$ to the nearest degree. **2**



By cosine rule in ΔAOB,

$$\cos \angle AOB = \frac{24^2 + 35^2 - 40^2}{2(24)(35)}$$ ✓

$$= \frac{67}{560} \quad (\doteqdot 0.119643)$$

$\therefore \angle AOB = 83°$ (nearest degree) ✓

(ii) Calculate the area of the planned national park, correct to the nearest square km. **2**

$A = \frac{1}{2}(24)(35)\sin 83°$ ✓

$= 416.86938\ldots$

$\therefore A = 417 \text{ km}^2$ (nearest km^2) ✓

(iii) What is the bearing of O from B ? **1**

(see diagram in (i)) :

Bearing $= 360 - [180 - (248 - 83)]$

$\therefore = 345°T$ ✓

Question 16 (5 marks)

(i) Find the integral $\int \frac{10x}{x^2-8}\,dx$ **2**

$I = 5\int \frac{2x}{x^2-8}\,dx$

(1 off for no constant or no abs. value).

$\therefore I = 5\ln|x^2-8| + c$ ✓✓

(ii) Find the exact value of $\int_2^6 2^{x-1}\,dx$ **3**

$I = \left[\frac{2^{x-1}}{\ln 2}\right]_2^6$ ✓

$= \frac{1}{\ln 2}(2^5 - 2^1)$ ✓

$\therefore I = \frac{30}{\ln 2}$ ✓

Question 17 (2 marks)

Find $\frac{d}{dx}\ln\frac{x+1}{x-2}$ **2**

$= \frac{d}{dx}\left[\ln(x+1) - \ln(x-2)\right]$ ✓

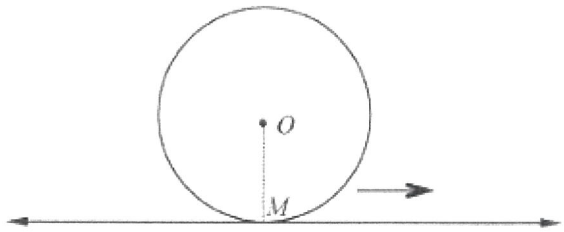
$= \frac{1}{x+1} - \frac{1}{x-2}$

$= \frac{x-2-x-1}{(x+1)(x-2)} = \frac{-3}{(x+1)(x-2)}$ ✓

Question 18 (4 marks)

A train wheel of radius 40 cm and centre *O* rolls along a horizontal track (as shown below).

M is a point on the wheel where the wheel touches the track before it starts to roll.



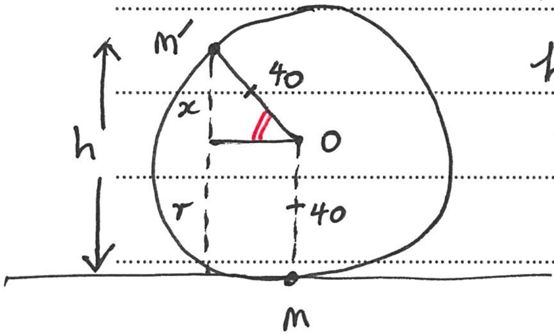
(i) Through what angle does *M* rotate about *O* (in radians) after the wheel rolls 1 metre ? **1**

$l = r\theta$

$\therefore$ 100 cm = 40 cm $\times \theta$

$\therefore \theta = \frac{5}{2}$ radians

(ii) After the wheel rolls 1 metre, what is the vertical height of *M* above the track? (answer to the nearest cm) **3**



M′ is where M rolls to, and

h = height of M′ above track.

The angle shown in the triangle

is $\frac{5}{2} - \frac{\pi}{2} = \frac{5-\pi}{2}$ radians.

Then $\frac{x}{40} = \sin\frac{5-\pi}{2}$ $\therefore x = 40\sin\frac{5-\pi}{2}$

$\therefore h = 40 + 40\sin\left(\frac{5-\pi}{2}\right)$

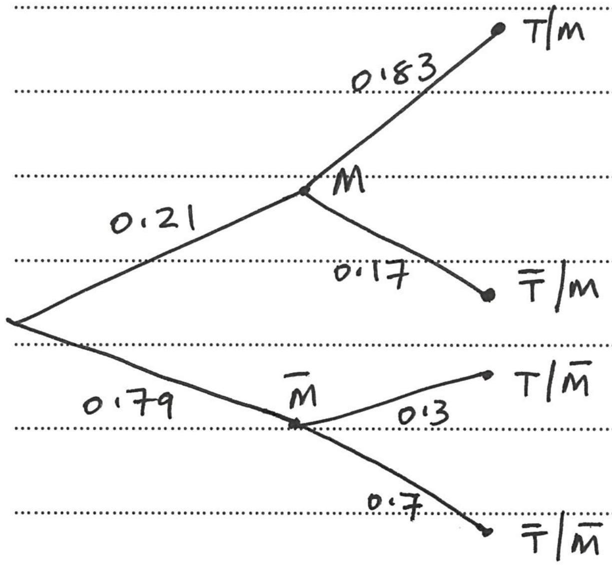
$\doteqdot 72.0457$

$\therefore h = 72$ cm (nearest cm)

Question 19 (5 marks)

In Glenrowan, the probability that it rains on any Monday is $0 \cdot 21$. If it rains on Monday, then the probability that it rains on Tuesday is $0 \cdot 83$. If it does not rain on Monday, then the probability of rain on Tuesday is $0 \cdot 3$.

(i) Draw a probability tree diagram to show this scenario. **2**



'M' = rain on Monday
$\bar{M}$ = not rain on Mon.
T = rain on Tuesday
$\bar{T}$ = not rain Tues.

(ii) Calculate the probability that it rains on a Tuesday in any given week. **1**

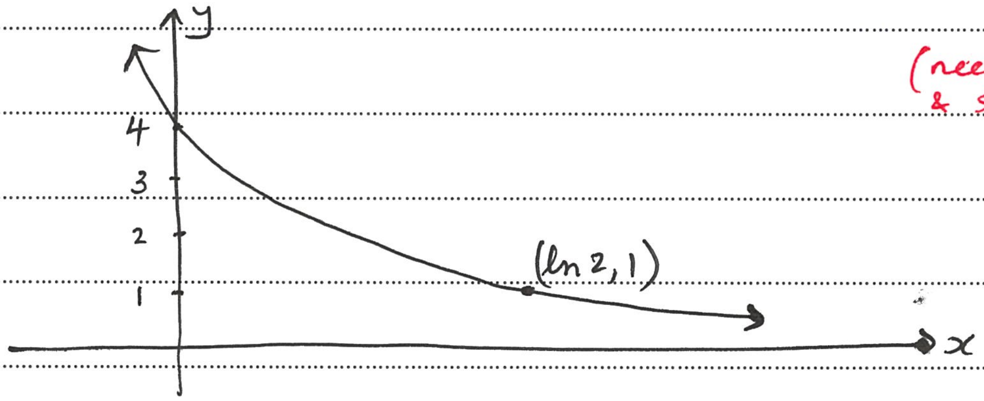
$$P(\text{rains on Tues.}) = (0 \cdot 21)(0 \cdot 83) + (0 \cdot 79)(0 \cdot 3) = \frac{4113}{10\,000} = 0 \cdot 4113$$

(iii) Given that it rains on a Tuesday, find the chance it did not rain on the Monday. **2**

$$P(\bar{M} | T) = \frac{P(\bar{M} \cap T)}{P(T)} = \frac{(0 \cdot 79)(0 \cdot 3)}{0 \cdot 4113} = \frac{790}{1371} \doteq 0 \cdot 57622\ldots = 0 \cdot 576 \text{ (3 d.p.)}$$

Question 20 (6 marks)

(a) Sketch the curve $y = 4e^{-2x}$ **2**



(needs y-int & some other point)

✓✓

(b) Consider the series $2e^{x} + 8e^{-x} + 32e^{-3x} + \cdots$

(i) Show that this series is geometric **1**

$\frac{T_2}{T_1} = \frac{8e^{-x}}{2e^{x}} = 4e^{-2x}$, $\frac{T_3}{T_2} = \frac{32e^{-3x}}{8e^{-x}} = 4e^{-2x} = \frac{T_2}{T_1}$ ✓

(∴ it's a GP with $a = 2e^{x}$, $r = 4e^{-2x}$)

(ii) Find the values of x for which this series has a limiting sum. **2**

We need $-1 < 4e^{-2x} < 1$ for limiting sum.

From the graph in (a), $-1 < 4e^{-2x} < 1$ for $x > \ln 2$.

ie limit exists when $x > \ln 2$ ✓✓

(ii) Find the limiting sum of this series in terms of x. **1**

$S_\infty = \frac{a}{1-r} = \frac{2e^{x}}{1-4e^{-2x}}$ ✓

Question 21 (6 marks)

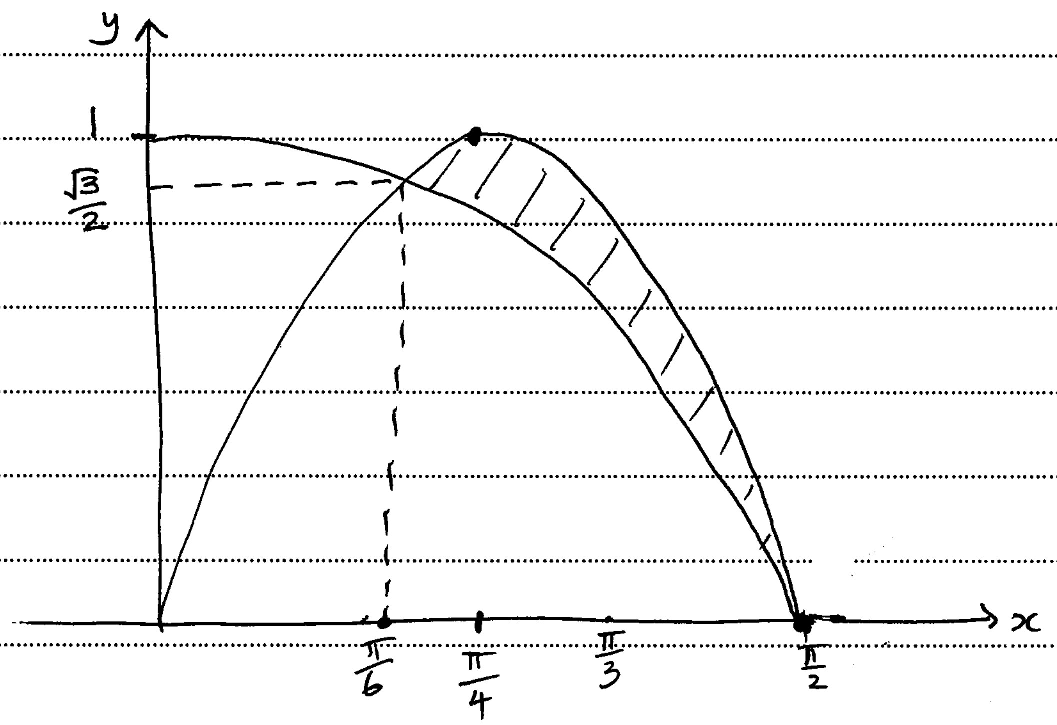
(i) Show that the curves $y = \sin 2x$ and $y = \cos x$ meet at the point where $x = \frac{\pi}{6}$. **1**

When $x = \pi/6$, $\sin 2x = \sin\frac{\pi}{3} = \sqrt{3}/2$

When $x = \frac{\pi}{6}$, $\cos x = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\therefore$ curves meet at $\left(\pi/6, \sqrt{3}/2\right)$ ✓

(ii) Sketch, on the same set of axes, the curves $y = \sin 2x$ and $y = \cos x$ in the domain $0 \le x \le \frac{\pi}{2}$ and shade the region bounded only by those curves. **2**



✓✓

(iii) Find the area of this shaded region. **3**

$$A = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin 2x - \cos x)\, dx$$

$$= \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$$

$$= \left(-\tfrac{1}{2}\cos \pi - \sin \tfrac{\pi}{2}\right) - \left(-\tfrac{1}{2}\cos \tfrac{\pi}{3} - \sin \tfrac{\pi}{6}\right)$$

$$= \left(\tfrac{1}{2} - 1\right) - \left(-\tfrac{1}{4} - \tfrac{1}{2}\right)$$

$$= \tfrac{1}{4}$$

$$\therefore A = \tfrac{1}{4}\, u^2$$

Question 22 (2 marks)

Evaluate $\sum_{r=1}^{63} \frac{1}{\sqrt{r+1}+\sqrt{r}}$ **2**

$$= \sum_{r=1}^{63} \left(\frac{1}{\sqrt{r+1}+\sqrt{r}} \times \frac{\sqrt{r+1}-\sqrt{r}}{\sqrt{r+1}-\sqrt{r}}\right)$$

$$= \sum_{r=1}^{63} \left(\sqrt{r+1} - \sqrt{r}\right)$$

$$= (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + (\sqrt{5}-\sqrt{4}) + \ldots$$

$$\ldots + (\sqrt{63}-\sqrt{62}) + (\sqrt{64}-\sqrt{63})$$

$$= -1 + \sqrt{64} = 7$$

Question 23 (5 marks)

A tank contains 200 litres of petrol. More petrol is pumped into the tank for 20 minutes, until it is full.

The volume flow rate R of petrol, in litres per minute, is given by

$$R = 4\,(20 - t)$$

(i) Find a formula for the volume V of petrol in the tank after t minutes, where $t \leq 20$. **2**

$R = \frac{dV}{dt} = 4(20-t) = 80 - 4t$

$\therefore V = \int (80 - 4t)\,dt = 80t - 2t^2 + C$ ✓

at $t=0$, $V=200$ $\therefore 200 = 0 - 0 + C$ $\therefore C = 200$

$\therefore V = 80t - 2t^2 + 200$ ✓

(ii) How many litres of petrol were in the tank when it was full? **1**

'Full' $\Rightarrow \frac{dV}{dt} = 0$ $\therefore 4(20-t) = 0$ $\therefore t = 20$

At $t = 20$, $V = 80(20) - 2(20^2) + 200 = 1000\text{ L}$ ✓

(iii) How long did it take to half fill the tank? (answer in minutes) **2**

We have $80t - 2t^2 + 200 = 500$

$2t^2 - 80t + 300 = 0$

$t^2 - 40t + 150 = 0$

$\therefore t = \frac{40 \pm \sqrt{1600 - 600}}{2} = \frac{40 \pm \sqrt{1000}}{2}$ ✓

So $t = 35.811$ or $t = 4.1886$. ✓

But tank full at $t = 20$, $\therefore$ $t = 4.1886$ minutes. (4 mins. 11 secs.)

Question 24 (2 marks)

If $log_a(xy^3) = 1$ and $log_a(x^2y) = 1$ what is the value of $log_a(xy)$? 2

We have $\log_a x + 3\log_a y = 1$ — ①

$2\log_a x + \log_a y = 1$ — ②

$2\times$① → $2\log_a x + 6\log_a y = 2$ --- ③

③−② → $5\log_a y = 1 \quad \therefore \log_a y = \frac{1}{5}$

So $\therefore \log_a x = 1 - 3\log_a y = 1 - \frac{3}{5} = \frac{2}{5}$

$\therefore \log_a(xy) = \log_a x + \log_a y = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$ ✓✓

Question 25 (3 marks)

Isis and Layla catch a train to school each day but they get on at different stations.

The probability that Isis gets a seat each morning is $\frac{1}{8}$.

The probability that Layla gets a seat each morning is $\frac{1}{21}$.

What is the probability that, on the next three mornings, at least one girl gets a seat? 3

$P(\text{I gets seat}) = \frac{1}{8} \quad \therefore P(\text{I doesn't get seat}) = \frac{7}{8}$

$P(\text{L gets seat}) = \frac{1}{21} \quad \therefore P(\text{L not get seat}) = \frac{20}{21}$

$\therefore P(\text{neither gets seat}) = \frac{7}{8} \times \frac{20}{21} = \frac{5}{6}$. ✓

$\therefore P(\text{neither gets seat, 3 days in a row}) = \left(\frac{5}{6}\right)^3 = \frac{125}{216}$ ✓

$\therefore P(\text{at least 1 gets a seat in next 3 days}) = 1 - \frac{125}{216}$

$= \frac{91}{216}$ ✓

Question 26 (8 marks)

Consider the curve $y = x^3 + 3x^2 - 9x - 5$.

(i) Find any stationary points and determine their nature. **4**

$y' = 3x^2 + 6x - 9$ $\qquad$ $y'' = 6x + 6$

$= 3(x^2 + 2x - 3)$ $\qquad$ $\therefore y'' = 6(x+1)$

$\therefore y' = 3(x+3)(x-1)$ ✓

For stat. pts, $y' = 0$

$\therefore x = -3$ or $x = 1$

$y = 22$ $\qquad$ $y = -10$ ✓

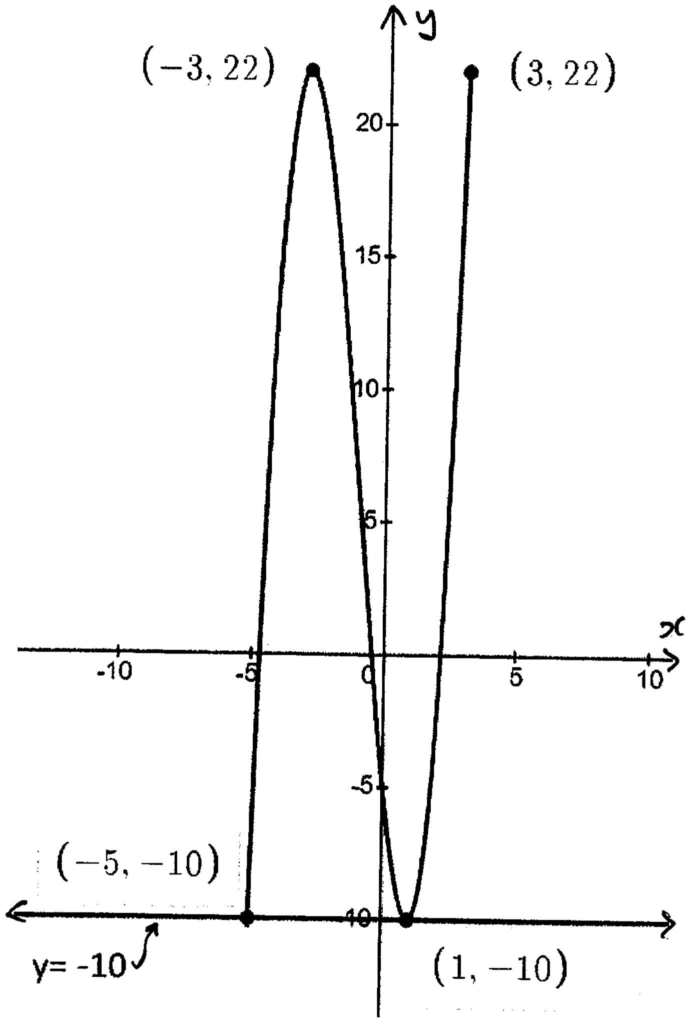
at $(-3, 22)$, $y'' = 6(-3+1) = -12 < 0$

$\therefore (-3, 22)$ is maximum turning point. ✓

at $(1, -10)$, $y'' = 6(1+1) = 12 > 0$

$\therefore (1, -10)$ is minimum turning point. ✓

(ii) Sketch the curve in the domain $-5 \leq x \leq 3$ (inflection and x-intercepts are not required) **2**



(iii) By drawing an appropriate line on your graph, or otherwise, solve

$$x^3 + 3x^2 - 9x + 5 = 0$$ **2**

$x^3 + 3x^2 - 9x + 5 = 0$

$\therefore x^3 + 3x^2 - 9x - 5 = -10$

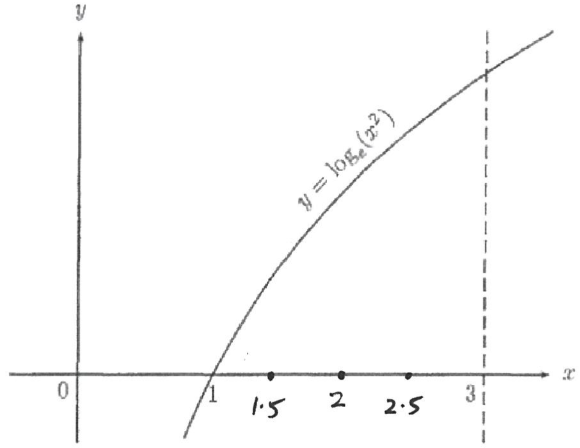
Since the cubic we sketched meets $y = -10$

at $x = -5$ & $x = 1$

$\therefore$ Solution is $x = -5, 1$

Question 27 (6 marks)

The graph of the function $y = log_e(x^2)$ is shown below:



(i) Use the Trapezoidal Rule with 5 function values to approximate $\int_1^3 log_e(x^2)\,dx$ to 3 significant figures, and explain why this approximation underestimates the value of the integral. **3**

x	1	1·5	2	2·5	3
$\ln(x^2)$	0	0·8109	1·3863	1·8326	2·1972

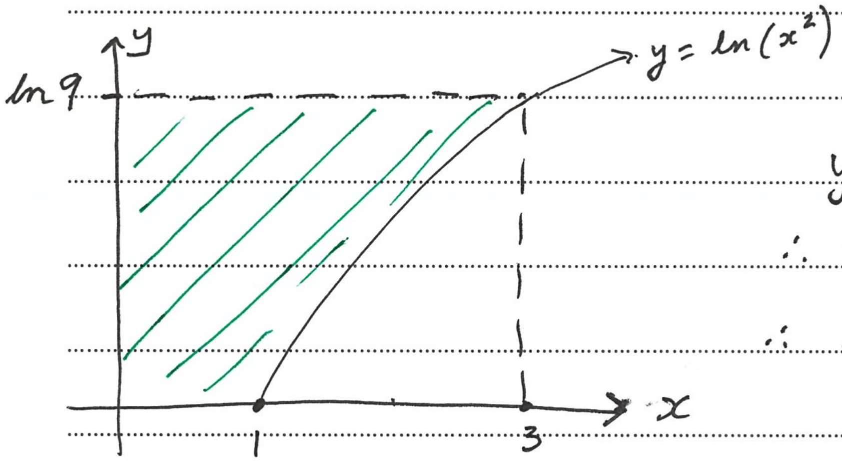
$$\therefore \int_1^3 \ln(x^2)\,dx \doteqdot \frac{3-1}{2\times4}\left[0 + 2\cdot1972 + 2\left(0\cdot8109 + 1\cdot3863 + 1\cdot8326\right)\right]$$

$$\doteqdot 2\cdot5642$$

$$\therefore = 2\cdot564 \quad \text{(3 sig. figs).}$$

Because $y = \ln(x^2)$ is concave down, each trapezium's sloping edge lies under the curve. The estimate is thus short by the amount between curve & edges.

(ii) Find $\int_0^{\ln 9} e^{\frac{y}{2}}\,dy$ and hence find the exact value of $\int_1^3 log_e(x^2)\,dx$ **3**



$y = \log_e x^2$

$\therefore x^2 = e^y$

$\therefore x = e^{\frac{y}{2}}$ (as $x > 0$)

Area under curve = area rectangle − shaded area

$= \ln 9 \times 3 - \int_0^{\ln 9} e^{\frac{y}{2}}\,dy$

$= 3\ln 9 - \left[2e^{\frac{y}{2}}\right]_0^{\ln 9}$

$= 3\ln 9 - 2e^{\frac{2\ln 3}{2}} + 2e^0$

$= 3\ln 9 - 6 + 2 = 3\ln 9 - 4$

Question 28 (2 marks)

Find $f'(x)$ if it is known that $\frac{d}{dx}[f(2x)] = x^2$ **2**

We have $\frac{d}{dx}[f(2x)] = 2f'(2x) = x^2$

$\therefore f'(2x) = \frac{1}{2}x^2$

$= \frac{1}{8}(2x)^2$

i.e. $f'(u) = \frac{1}{8}u^2$ if we let $u = 2x$.

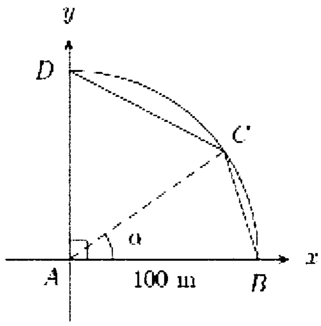
That is, $f'(x) = \frac{1}{8}x^2$.

Question 29 (8 marks)

ABCD is a quadrilateral inscribed in a quarter of a circle centred at *A*, radius 100 m.

The points *B* and *D* lie on the *x* and *y* axes and the point *C* moves on the circle such that

$\angle CAB = \alpha$ as shown in the diagram below.



(i) Solve the equation $\sin(x + 15°) = \cos 24°$ **1**

$\sin(x+15) = \cos 24$

$= \sin(90 - 24) = \sin 66$

$\therefore x + 15 = 66 \quad \therefore x = 51°$ ✓

(ii) Show that the area of the quadrilateral ABCD can be expressed as **3**

$$A = 5000\,(\sin\alpha + \cos\alpha)$$

$A_{\Delta CAB} = \frac{1}{2} \cdot 100 \cdot 100 \sin\alpha = 5000 \sin\alpha$ ✓

$A_{\Delta CAD} = \frac{1}{2} \cdot 100 \cdot 100 \cdot \sin\left(\frac{\pi}{2} - \alpha\right) = 5000 \cos\alpha$ ✓

$\therefore A_{ABCD} = 5000 \sin\alpha + 5000 \cos\alpha$

$= 5000(\sin\alpha + \cos\alpha)$ ✓

(iii) Using calculus, show that the maximum area of this quadrilateral is $5000\sqrt{2}\, m^2$ **4**

$A = 5000(\sin\alpha + \cos\alpha)$

$\therefore \frac{dA}{d\alpha} = 5000(\cos\alpha - \sin\alpha)$ ✓

For stat. points, $\frac{dA}{d\alpha} = 0$ $\therefore \cos\alpha - \sin\alpha = 0$

$\therefore \tan\alpha = 1$

$\therefore \alpha = \frac{\pi}{4}$ ✓

$\frac{d^2A}{d\alpha^2} = 5000(-\sin\alpha - \cos\alpha)$

$= 5000\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$

$= -5000\sqrt{2} < 0$

$\therefore$ maximum area when $\alpha = \pi/4$ ✓

Thus max. area $= 5000\left(\sin\frac{\pi}{4} + \cos\pi/4\right)$

$= 5000\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$

$= 5000\sqrt{2}\ m^2.$ ✓

Question 30 (3 marks)

Let $f(x) = x^2 - 4x$.

Sketch the graph of $y = 2f(1-x) + 6$, showing the location of the vertex and the intercepts. **3**

Since $2f(1-x)+6 = 2f(-(x-1)) + 6$, our graph will be $f(x)$ transformed as follows: – reflect in y-axis; shift Right by 1; dilate by factor 2 vertically; shift up by 6.

Hence new vertex is $(2,-4) \to (-1,-2)$

Easiest is just to transform algebraically :–

$y = 2f(1-x)+6 = 2\left[(1-x)^2 - 4(1-x)\right] + 6$

$\therefore y = 2\left[1-2x+x^2-4+4x\right]+6 = 2(x^2+2x-3)+6$

$\therefore y = x^2 + 2x = x(x+2)$

↳ ∴ x-ints are 0, −2

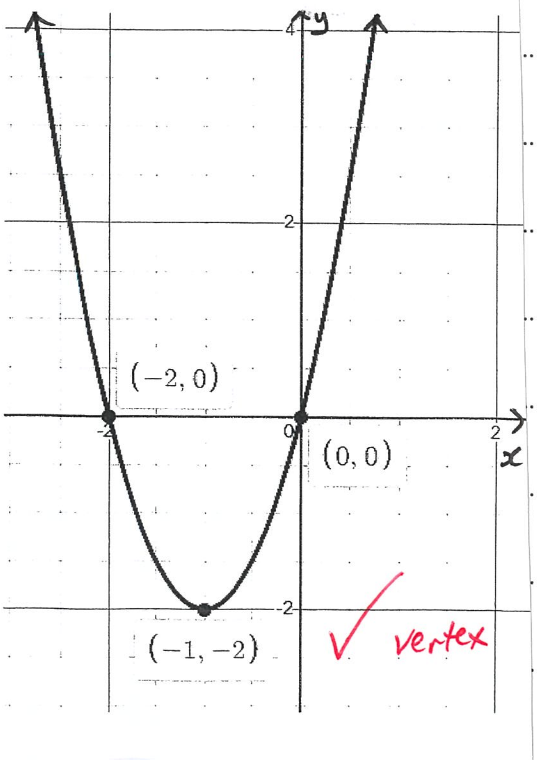
& axis is $x=-1$

∴ $V = (-1, -2)$

you should get this:

✓ some useful working

✓ intercept(s)

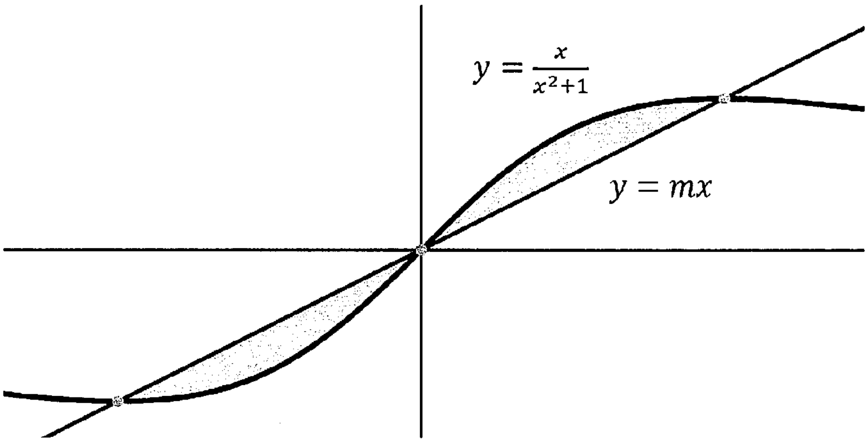


$y = 2f(1-x)+6$.

Question 31 (5 marks)

The function $y = \frac{x}{x^2+1}$ is shown below. It is an odd function.

(i) For what values of m do the line $y = mx$ and the curve $y = \frac{x}{x^2+1}$ enclose a region? **2**



For a region, we need 3 solutions to

$$\frac{x}{x^2+1} = mx$$

i.e. 3 solutions to $x = mx^3 + mx$

We see $mx^3 + (m-1)x = 0$

$$x\left(mx^2 + (m-1)\right) = 0$$

So one solution will be $x = 0$.

For the other 2 solutions we need $x^2 = \frac{1-m}{m} = \frac{1}{m} - 1$

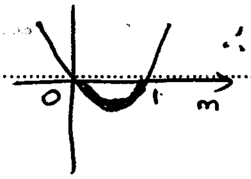
i.e. $x = \pm\sqrt{\frac{1}{m} - 1}$

For 2 solutions, $\frac{1}{m} - 1 > 0 \quad \therefore \frac{1}{m} > 1$

$\therefore$ $0 < m < 1$ to make a region. ✓✓

[ALT: In $mx^2 + (m-1) = 0$ we need $\Delta > 0$

ie $-4m(m-1) > 0 \quad \therefore 4m(m-1) < 0$



$\therefore 0 < m < 1$].

(ii) Find the area of the region (in terms of *m*). **3**

Since $y = \frac{x}{x^2+1}$ and $y = mx$ are both odd,

$\therefore$ shaded area $= 2 \times \int_0^{\sqrt{\frac{1}{m}-1}} \left(\frac{x}{x^2+1} - mx \right) dx$ ✓

$= 2 \times \left[\frac{1}{2} \ln|x^2+1| - \frac{mx^2}{2} \right]_0^{\sqrt{\frac{1}{m}-1}}$ ✓

$= \ln\left|\frac{1}{m} - 1 + 1\right| - m\left(\frac{1}{m} - 1\right) - (\ln 1 - 0)$

$= \ln\left|\frac{1}{m}\right| + m - 1 \ u^2$ ✓

Question 32 (5 marks)

(i) Show that $e^x \geq 1 + x$ if $x \geq 0$. **2**

(Hint: show that the function $f(x) = e^x - (1 + x)$ is increasing for $x \geq 0$.)

Let $f(x) = e^x - 1 - x$.

Then $f(0) = e^0 - 1 - 0 = 0$

Also, for $x > 0$ we have $f'(x) = e^x - 1 > 0$.

So, since $f(0) = 0$ and $f(x)$ is increasing for $x \geq 0$,

we must have $f(x) \geq 0$ for $x \geq 0$.

i.e $e^x - 1 - x \geq 0$

i.e. $e^x \geq 1 + x$. ✓✓

(ii) Deduce that $\frac{4}{3} \leq \int_0^1 e^{x^2}\, dx \leq e$ **3**

For $0 \leq x \leq 1$, $x^2 \leq x$ $\therefore e^{x^2} \leq e^x$ (as e^x is an increasing function).

So, from (i), $\therefore 1 + x^2 \leq e^{x^2} \leq e^x$ ✓

$\therefore \int_0^1 (1+x^2)\, dx \leq \int_0^1 e^{x^2}\, dx \leq \int_0^1 e^x\, dx$ ✓

$\therefore \left[x + \frac{x^3}{3}\right]_0^1 \leq \int_0^1 e^{x^2}\, dx \leq \left[e^x\right]_0^1$

$\therefore \left(1 + \frac{1}{3}\right) - (0 - 0) \leq \int_0^1 e^{x^2}\, dx \leq e^1 - e^0$ ✓

i.e. $\frac{4}{3} \leq \int_0^1 e^{x^2}\, dx \leq e - 1 < e$, as required.

ENDS